

Pentagon relation for the quantum dilogarithm and quantized $\mathcal{M}_{0,5}^{\text{cyc}}$

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To the memory of Sasha Reznikov

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1 Introduction

Let $\hbar > 0$. The quantum dilogarithm function is given by the following integral:

$$\Phi^{\hbar}(z) := \exp\left(-\frac{1}{4} \int_{\Omega} \frac{e^{-ipz}}{\text{sh}(\pi p) \text{sh}(\pi \hbar p)} \frac{dp}{p}\right), \quad \text{sh}(p) = \frac{e^p - e^{-p}}{2}.$$

Here Ω is a path from $-\infty$ to $+\infty$ making a little half circle going over the zero. So the integral is convergent. It goes back to Barnes [Ba], and appeared in many papers during the last 30 years: [Bax], [Sh], [Fad1], The function $\Phi^{\hbar}(z)$ enjoys the following properties (cf. [FG3], Section 4):

- The function $\Phi^{\hbar}(z)$ is meromorphic. Its zeros are simple zeros in the upper half plane at the points

$$\{\pi i((2m-1) + (2n-1)\hbar) | m, n \in \mathbb{N}\}, \quad \mathbb{N} := \{1, 2, \dots\}. \quad (1)$$

Its poles are simple poles, located in the lower half plane, at the points

$$\{-\pi i((2m-1) + (2n-1)\hbar) | m, n \in \mathbb{N}\}. \quad (2)$$

- The function $\Phi^{\hbar}(z)$ is characterized by the following difference relations. Let $q := e^{\pi i \hbar}$ and $q^{\vee} := e^{\pi i / \hbar}$. Then

$$\Phi^{\hbar}(z + 2\pi i \hbar) = \Phi^{\hbar}(z)(1 + qe^z), \quad \Phi^{\hbar}(z + 2\pi i) = \Phi^{\hbar}(z)(1 + q^{\vee}e^{z/\hbar}), \quad (3)$$

- One has $|\Phi^{\hbar}(z)| = 1$ when z is on the real line.

- It is related in several ways to the dilogarithm, e.g. its asymptotic expansion when $\hbar \rightarrow 0$ is

$$\Phi^{\hbar}(z) \sim \exp\left(\frac{L_2(e^z)}{2\pi i \hbar}\right), \quad \text{where } L_2(x) := \int_0^x \log(1+t) \frac{dt}{t}$$

is a version of the Euler's dilogarithm function.

When \hbar is a complex number with $\text{Im } \hbar > 0$, there is an infinite product expansion

$$\Phi^{\hbar}(z) = \frac{\Psi^q(e^z)}{\Psi^{1/q^\vee}(e^{z/\hbar})}, \quad \text{where } \Psi^q(x) := \prod_{a=1}^{\infty} (1 + q^{2a-1}x)^{-1}.$$

The function $\Phi^{\hbar}(z)$ provides an operator $K : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, defined as a rescaled Fourier transform followed by the operator of multiplication by the quantum dilogarithm $\Phi^{\hbar}(x)$.

$$Kf(z) := \int_{-\infty}^{\infty} f(x) \Phi^{\hbar}(x) \exp\left(\frac{-xz}{2\pi i \hbar}\right) dx.$$

Since $|\Phi^{\hbar}(x)| = 1$ on the real line, $2\pi\sqrt{\hbar}K$ is unitary.

Theorem 1.1 $(2\pi\sqrt{\hbar}K)^5 = \lambda \cdot \text{Id}$, where $|\lambda| = 1$.

In the quasiclassical limit it gives Abel's five term relation for the dilogarithm.

The pentagon relation for the simpler version $\Psi^q(x)$ of the quantum dilogarithm was discovered in [FK]. A similar pentagon relation for the function $\Phi^{\hbar}(z)$, which is equivalent to Theorem 1.1, was suggested in [Fad1] and proved, using different methods, in [Wo] and [FKV]. Theorem 1.1 was formulated in [CF]. However the argument presented there as a proof has a significant problem, which put on hold the program of quantization of Teichmüller spaces.

In this paper we show that the operator K is a part of a much more rigid structure, called the *quantized moduli space* $\mathcal{M}_{0,5}^{\text{cyc}}$ – this easily implies Theorem 1.1.

Namely, consider the algebra generated by operators of multiplication by e^x and $e^{x/\hbar}$ and shifts by $2\pi i$ and $2\pi i \hbar$, acting as unbounded operators in $L^2(\mathbb{R})$. We use a remarkable subalgebra \mathbf{L} of this $*$ -algebra, and introduce a *Schwartz space* $S_{\mathbf{L}} \subset L^2(\mathbb{R})$, defined as the common domain of the operators from \mathbf{L} . It comes with a natural topology. Our main result, Theorem 2.6, tells that the operator K preserves the space $S_{\mathbf{L}}$, and the conjugation by K intertwines an order 5 automorphism γ of the algebra \mathbf{L} , see Fig. 1. This characterises the operator K up to a constant. The proof uses analytic properties of the space $S_{\mathbf{L}}$ developed in Theorem 2.3. Theorem 2.6 easily implies Theorem 1.1.

$$\begin{array}{ccc} S_{\mathbf{L}} & \xrightarrow{K} & S_{\mathbf{L}} \\ \circlearrowleft & & \circlearrowleft \\ \mathbf{L} & \xrightarrow{\gamma} & \mathbf{L} \end{array} \quad \begin{array}{l} K^5 = c \text{ Id} \\ \gamma^5 = \text{Id} \end{array}$$

Figure 1: Quantized moduli space $\mathcal{M}_{0,5}^{\text{cyc}}$.

We define a space of *distributions* $S_{\mathbf{L}}^*$ as the topological dual to $S_{\mathbf{L}}$. So there is a Gelfand triple $S_{\mathbf{L}} \subset L_2(\mathbb{R}) \subset S_{\mathbf{L}}^*$. The operator K acts by its automorphisms. It would be interesting to calculate it on some distributions explicitly.

The story is similar in spirit to the Fourier transform theory developed using the algebra of polynomial differential operators:

The Fourier transform $\langle - \rangle$ The operator K .

The algebra \mathbf{D} of polynomial differential operators $\langle - \rangle$ The algebra \mathbf{L} of difference operators.

The automorphism φ of \mathbf{D} given by $ix \rightarrow d/dx$, $d/dx \rightarrow -ix$ $\langle - \rangle$ The automorphism γ of \mathbf{L} .

The classical Schwartz space $\langle - \rangle$ The Schwartz space $S_{\mathbf{L}}$.

Let $\mathcal{M}_{0,5}^{\text{cyc}} \subset \overline{\mathcal{M}}_{0,5}$ be the moduli space of configurations of 5 cyclically ordered points on \mathbb{P}^1 , where we do not allow the neighbors to collide. It carries an atlas consisting of 5 coordinate systems, providing $\mathcal{M}_{0,5}^{\text{cyc}}$ with a structure of the cluster \mathcal{X} -variety of type A_2 . The algebra \mathbf{L} is isomorphic to the algebra of regular functions on the modular double of the non-commutative q -deformation of the cluster \mathcal{X} -variety. The automorphism γ corresponds to a cyclic shift acting on configurations of points.

The triple $(\mathbf{L}, S_{\mathbf{L}}, \gamma)$, see Fig. 1, is called the quantized moduli space $\mathcal{M}_{0,5}^{\text{cyc}}$.

The results of this paper admit a generalization to a cluster set-up, where the role of the automorphism γ plays the cluster mapping class group. In particular this gives a definition of quantized higher Teichmüller spaces, and allows to state precisely the modular functor property of the latter.¹

The structure of the paper. In Section 2.1 we recall the cluster \mathcal{X} -variety of type A_2 [FG2]. In Section 3 we identify it with $\mathcal{M}_{0,5}^{\text{cyc}}$. This clarifies formulas in Section 2.1-2.2. In Section 2.2 we recall a collection of regular functions on our cluster \mathcal{X} -variety. Theorem 3.2 tells that they form a basis in the space of regular functions, and in particular closed under multiplication. We introduce a q -deformed version of this basis/algebra. Its tensor product with a similar algebra for q^\vee is the algebra \mathbf{L} . In Sections 2.3-2.5 we prove our main results.

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2 Quantized moduli space $\mathcal{M}_{0,5}^{\text{cyc}}$

2.1 Cluster varieties of type A_2

The cluster \mathcal{X} -variety is glued from five copies of $\mathbb{C}^* \times \mathbb{C}^*$, so that i -th copy is glued to $(i+1)$ -st (indexes are modulo 5) by the map acting on the coordinate functions as follows:²

$$\gamma_X^* : X \mapsto Y^{-1}, \quad Y \mapsto (1+Y)X. \quad (4)$$

Similarly, the cluster \mathcal{A} -variety is glued from five copies of $\mathbb{C}^* \times \mathbb{C}^*$, so that i -th copy is glued to $(i+1)$ -th by the map acting on the coordinate functions as follows:

$$\gamma_A^* : (A, B) \mapsto ((1+A)B^{-1}, A). \quad (5)$$

¹In previous versions of quantization of Teichmüller spaces/cluster \mathcal{X} -varieties the pair $(\mathbf{L}, S_{\mathbf{L}})$ was missing, making the resulting notion rather flabby.

²We use a definition which differs slightly from the standard one, but delivers the same object.

(Accidentally, these two cluster varieties are canonically isomorphic).

The fifth degree of each of these maps is the identity. Thus the map identifying the i -th copy of $\mathbb{C}^* \times \mathbb{C}^*$ with the $(i+1)$ -st one in the standard way is an automorphism of order 5 acting on the \mathcal{X} - and \mathcal{A} -varieties. We denote it by γ .

Recall the tropical semifield \mathbb{Z}^t . It is the set \mathbb{Z} with the operations of addition $a \oplus b := \max\{a, b\}$, and multiplication $a \otimes b := a + b$. The set $\mathcal{A}(\mathbb{Z}^t)$ of \mathbb{Z}^t -points of the \mathcal{A} -variety is defined by gluing the five copies of \mathbb{Z}^2 via the tropicalizations of the map (5). The map γ acts on the tropical \mathcal{A} -space by

$$\gamma_a : (a, b) \mapsto (\max(a, 0) - b, a), \quad \gamma^5 = \text{Id}.$$

There are five cones in the tropical \mathcal{A} -space, shown on Fig. 2. The map γ shifts them cyclically counterclockwise. It is a piecewise linear map, whose restriction to each cone is linear.

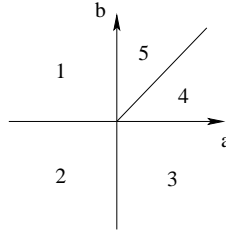


Figure 2: The five domains in the tropical \mathcal{A} -space.

2.2 The $*$ -algebra \mathbb{L}

The canonical basis for the cluster \mathcal{X} -variety of type A_2 . A rational function $F(X, Y)$ is a *universally positive Laurent polynomial* on \mathcal{X} , if $(\gamma_X^*)^i F(X, Y)$ is a Laurent polynomial with positive integral coefficients for every i . Equivalently, it belongs to the intersection of the ring of regular functions on the scheme \mathcal{X} over \mathbb{Z} with the semifield of rational functions with positive integral coefficients. There is a canonical γ -equivariant map, defined in Section 4 of [FG2]:³

$$\mathbb{I}_{\mathcal{A}} : \mathcal{A}(\mathbb{Z}^t) \longrightarrow \text{The space of universally positive Laurent polynomials on } \mathcal{X},$$

$$\gamma_X^* \mathbb{I}_{\mathcal{A}}(\gamma_a(a, b)) = \mathbb{I}_{\mathcal{A}}(a, b), \quad (6)$$

given by:

$$\mathbb{I}_{\mathcal{A}}(a, b) = \begin{cases} X^a Y^b & \text{for } a \leq 0 \text{ and } b \geq 0 \\ \left(\frac{1+X}{XY}\right)^{-b} X^a & \text{for } a \leq 0 \text{ and } b \leq 0 \\ \left(\frac{1+X+XY}{Y}\right)^a \left(\frac{1+X}{XY}\right)^{-b} & \text{for } a \geq 0 \text{ and } b \leq 0 \\ ((1+Y)X)^b \left(\frac{1+X+XY}{Y}\right)^{a-b} & \text{for } a \geq b \geq 0 \\ Y^{b-a} ((1+Y)X)^a & \text{for } b \geq a \geq 0. \end{cases} \quad (7)$$

Or equivalently, showing that the leading monomial is always $X^a Y^b$:

$$\mathbb{I}_{\mathcal{A}}(a, b) = \begin{cases} X^a Y^b & \text{for } a \leq 0 \text{ and } b \geq 0 \\ X^a Y^b (1 + X^{-1})^{-b} & \text{for } a \leq 0 \text{ and } b \leq 0 \\ X^a Y^b (1 + X^{-1})^{-b} (1 + Y^{-1} + X^{-1} Y^{-1})^a & \text{for } a \geq 0 \text{ and } b \leq 0 \\ X^a Y^b (1 + Y^{-1})^b (1 + Y^{-1} + X^{-1} Y^{-1})^{a-b} & \text{for } a \geq b \geq 0 \\ X^a Y^b (1 + Y^{-1})^a & \text{for } b \geq a \geq 0. \end{cases}$$

³Observe that γ_X^* tells how the automorphism γ acts on functions, while γ_a tells the action on the tropical points.

One can easily verify that the formulae agree on the overlapping domains of values of a and b . The i -th row of (7) describes the restriction of the canonical map to the i -th cone.

The quantum \mathcal{X} -variety and the quantum canonical basis. Let T_q be the algebra generated over $\mathbb{Z}[q, q^{-1}]$ by $X^{\pm 1}, Y^{\pm 1}$, subject to the relation $q^{-1}XY - qYX = 0$. It is called the two dimensional quantum torus algebra. It has an involutive antiautomorphism $*$ such that

$$*q = q^{-1}, \quad *X = X, \quad *Y = Y.$$

Consider the following q -deformation of the $*$ -equivariant map γ :

$$\gamma_q^* : X \mapsto Y^{-1}, \quad Y \mapsto (1 + qY)X. \quad (8)$$

One checks that it is an order 5 automorphism of the fraction field of T_q . The quantum \mathcal{X} -space \mathcal{X}_q is nothing else but a pair (T_q, γ_q^*) .⁴

An element $F(X, Y)$ of the fraction field of T_q is a *universally positive Laurent polynomial* on \mathcal{X}_q if $(\gamma_q^*)^i F(X, Y)$ is a Laurent polynomial in X, Y, q with positive integral coefficients for every i .

Proposition 2.1 *There is a canonical γ -equivariant map*

$$\mathbb{I}_{\mathcal{A}}^q : \mathcal{A}(\mathbb{Z}^t) \longrightarrow \text{The space of universally positive Laurent polynomials on } \mathcal{X}_q.$$

Construction. It is obtained by multiplying each monomial in (7) by a (uniquely defined) power of q , making it $*$ -invariant. For example, the quantum canonical map on the first cone is given by

$$\mathbb{I}_{\mathcal{A}}^q(a, b) = q^{-ab} X^a Y^b, \quad a \leq 0, b \geq 0.$$

Then we can use (17), which is valid in the q -deformed version as well.⁵

Denote by \mathbb{L}_q the image of the map $\mathbb{I}_{\mathcal{A}}^q$. It is closed under multiplication. Set $\mathbf{L} := \mathbb{L}_q \otimes \mathbb{L}_{q^\vee}$.

2.3 The Schwartz space $S_{\mathbf{L}}$

Let $W \subset L^2(\mathbb{R})$ be the space of finite \mathbb{C} -linear combinations of the functions

$$e^{-ax^2/2+bx} P(x), \quad \text{where } P(x) \text{ is a polynomial in } x, \text{ and } a \in \mathbb{R}_{>0}, b \in \mathbb{C}. \quad (9)$$

Set

$$\begin{aligned} \widehat{X}(f)(x) &:= f(x + 2\pi i \hbar), \quad \hbar \in \mathbb{R}_{>0}, & \widehat{Y}(f)(x) &:= e^x f(x). \\ \widehat{X}^\vee(f)(x) &:= f(x + 2\pi i), \quad \hbar \in \mathbb{R}_{>0}, & \widehat{Y}^\vee(f)(x) &:= e^{x/\hbar} f(x). \end{aligned}$$

They are symmetric unbounded operators. They preserve W and satisfy, on W , relations

$$\begin{aligned} \widehat{X}\widehat{Y} &= q^2 \widehat{Y}\widehat{X}, \quad q := e^{\pi i \hbar}. \\ \widehat{X}^\vee \widehat{Y}^\vee &= (q^\vee)^2 \widehat{Y}^\vee \widehat{X}^\vee, \quad q^\vee := e^{\pi i / \hbar}. \end{aligned}$$

The second pair of operators commute with the first one. Therefore these operators provide an $*$ -representation of the algebra $T_q \otimes T_{q^\vee}$ in W .

Remark. Consider a smaller subspace $W_0 \subset W$, with $a = 1, b \in 2\pi i \hbar \mathbb{Z} + 2\pi i \mathbb{Z} + \mathbb{Z} + 1/\hbar \mathbb{Z}$ and $\deg(P) = 0$. Then acting on $e^{-x^2/2}$ we get an isomorphism of linear spaces $T_q \otimes T_{q^\vee} \cong W_0$.

In particular an element $A \in \mathbf{L}$ acts by an unbounded operator \widehat{A} in W .

⁴Alternatively, using a geometric language, the quantum \mathcal{X} -space \mathcal{X}_q is glued from five copies of the spectrum $\text{Spec}(T_q)$ of the quantum torus T_q , so that i -th copy is glued to $(i+1)$ -st along the map (8)

⁵We do not use the fact that Laurent q -polynomials in the basis have positive integral coefficients.

Definition 2.2 *The Schwartz space $S_{\mathbf{L}}$ for the $*$ -algebra \mathbf{L} is a subspace of $L^2(\mathbb{R})$ consisting of vectors f such that the functional $w \rightarrow (f, \hat{A}w)$ on W is continuous for the L_2 -norm.*

Denote by $(*, *)$ the scalar product in L_2 . The Schwartz space for the $*$ -algebra \mathbf{L} is the common domain of definition of operators from \mathbf{L} in $L^2(\mathbb{R})$. Indeed, since W is dense in $L^2(\mathbb{R})$, the Riesz theorem implies that for any $f \in S_{\mathbf{L}}$ there exists a unique $g \in L^2(\mathbb{R})$ such that $(g, w) = (f, \hat{A}w)$. We set $\hat{A}f := g$. Equivalently, let W^* be the algebraic linear dual to W . So $L^2(\mathbb{R}) \subset W^*$. Then

$$S_{\mathbf{L}} = \{v \in W^* | \hat{A}^*v \in L^2(\mathbb{R}) \text{ for any } A \in \mathbf{L}\} \cap L^2(\mathbb{R}).$$

The Schwartz space $S_{\mathbf{L}}$ has a natural topology given by seminorms

$$\rho_B(f) := \|Bf\|_{L_2}, \quad B \text{ runs through a basis in } \mathbf{L}.$$

The key properties of the Schwartz space $S_{\mathbf{L}}$ which we use below are the following.

Theorem 2.3 *The space W is dense in the Schwartz space $S_{\mathbf{L}}$.*

One can interpret Theorem 2.3 by saying that the $*$ -algebra \mathbf{L} is essentially self-adjoint in $L^2(\mathbb{R})$.

Proof.

Lemma 2.4 *For any $w \in W, s \in S_{\mathbf{L}}$, the convolution $s * w$ lies in $S_{\mathbf{L}}$.*

Proof. Set $T_\lambda f(x) := f(x - \lambda)$. Write

$$s * w(x) = \int_{-\infty}^{\infty} w(t)(T_t s)(x) dt.$$

For any seminorm ρ_B on $S_{\mathbf{L}}$ the operator $T_\lambda : (S_{\mathbf{L}}, \rho_B) \rightarrow (S_{\mathbf{L}}, \rho_B)$ is a bounded operator with the norm bounded by $e^{|\lambda|}$. Thus the operator $\int_{-\infty}^{\infty} w(t)T_t dt$ is a bounded operator on $(S_{\mathbf{L}}, \rho_B)$. This implies the Lemma.

Let $w_\varepsilon := (2\pi)^{-\frac{1}{2}}\varepsilon^{-1}e^{-\frac{1}{2}(x/\varepsilon)^2} \in W$ be a sequence converging as $\varepsilon \rightarrow 0$ to the δ -function at 0. Clearly one has in the topology of $S_{\mathbf{L}}$

$$\lim_{\varepsilon \rightarrow 0} w_\varepsilon * s = s(x). \quad (10)$$

Lemma 2.5 *For any $w \in W, s \in S_{\mathbf{L}}$, the Riemann sums for the integral*

$$s * w(x) = \int_{-\infty}^{\infty} s(t)w(x - t)dt = \int_{-\infty}^{\infty} s(t)T_t w(x)dt. \quad (11)$$

*converges in the topology of $S_{\mathbf{L}}$ to the convolution $s * w$.*

Proof. Let us show first that (11) is convergent in $L_2(\mathbb{R})$. The key fact is that a shift of $w \in W$ quickly becomes essentially orthogonal to w . More precisely, in the important for us case when $w = \exp(-ax^2/2 + bx)$, $a > 0$, (this includes any $w \in W_0$) we have

$$(w(x), T_\lambda w(x)) < C_w e^{-a\lambda^2/2 + (b-\bar{b})\lambda}. \quad (12)$$

Therefore in this case

$$\left(\int_{-\infty}^{\infty} s(t)T_t w(x)dt, \int_{-\infty}^{\infty} s(t)T_t w(x)dt \right)$$

$$\begin{aligned}
&\stackrel{(12)}{\leq} C_w \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(t_1-t_2)^2/2+(b-\bar{b})(t_1-t_2)} |s(t_1)s(t_2)| dt_1 dt_2 = \\
&C_w \int_{-\infty}^{\infty} e^{-a\lambda^2/2+(b-\bar{b})\lambda} \int_{-\infty}^{\infty} |s(t)s(t+\lambda)| dt d\lambda \leq C_w \|s\|_{L_2}^2 \int_{-\infty}^{\infty} e^{-a\lambda^2/2+(b-\bar{b})\lambda} d\lambda.
\end{aligned}$$

We leave the case of an arbitrary w to the reader: it is not used in the proof of the theorem.

The convergence with respect to the seminorm $\|Bf\|$ is proved by the same argument using the fact that $W_0 \subset W$ is stable under the algebra \mathbf{L} . The Theorem follows from Lemma 2.4, (10) and Lemma 2.5.

Remark. The same arguments show that the space W is dense in the space $S_{\mathbf{L}'}$ defined for any subalgebra \mathbf{L}' of \mathbf{L} .

2.4 The main result

Theorem 2.6 *The operator K preserves the Schwartz space $S_{\mathbf{L}}$. It intertwines the automorphism γ of the algebra \mathbf{L} , i.e. for any $A \in \mathbf{L}$ and $s \in S_{\mathbf{L}}$ one has*

$$K^{-1} \widehat{A} K s = \widehat{\gamma(A)} s. \quad (13)$$

Proof. We need the following key result.

Proposition 2.7 *For any $A \in \mathbf{L}$, $w \in W$ one has $K \widehat{\gamma(A)} w = \widehat{A} K w$. Therefore $\widehat{A} K w \in L^2(\mathbb{R})$.*

Proof. Let \mathbb{L}'_q be the space of Laurent q -polynomials F in X, Y such that $\gamma(F)$ is again a Laurent q -polynomial. The following elements belong to \mathbb{L}'_q :

$$X^a Y^m, \quad X^a Y^{-n} (1 + qX^{-1})(1 + q^3 X^{-1}) \dots (1 + q^{2n-1} X^{-1}), \quad a \in \mathbb{Z}, \quad m, n \geq 0. \quad (14)$$

Indeed, $\gamma(Y^{-n}) = ((1 + qY)X)^{-n} = X^{-n} \prod_{a=1}^n (1 + q^{(2a-1)} Y)^{-1}$.

Lemma 2.8 (i) *The monomials (14) span the space \mathbb{L}'_q .*

(ii) *For every $A \in \mathbb{L}'_q \otimes \mathbb{L}'_{q^\vee}$, $w \in W$ one has $K \widehat{\gamma(A)} w = \widehat{A} K w$.*

Lemma 2.8 implies Proposition 2.7. Thanks to the very definition $\mathbb{L}_q \subset \mathbb{L}'_q$. So the part (ii) of the Lemma includes the commutation relation from Proposition 2.7.

Proof of Lemma 2.8. (i) is obvious.

(ii) Let us prove first the following three basic identities:

$$K(1 + q\widehat{Y})\widehat{X}w = \widehat{Y}Kw; \quad K\widehat{Y}^{-1}w = \widehat{X}Kw; \quad K\widehat{X}^{-1}w = \widehat{Y}^{-1}(1 + q\widehat{X}^{-1})Kw. \quad (15)$$

The general case follows from this. To see this, observe that if $A_1, A_2 \in \mathbf{L}$ and $K\widehat{A}_i w = \widehat{\gamma^{-1}(A_i)} Kw$ for $i = 1, 2$, then, since $\widehat{A}_2 w \in W$, one has

$$K\widehat{A}_1 \widehat{A}_2 w = \widehat{\gamma^{-1}(A_1)} K\widehat{A}_2 w = \widehat{\gamma^{-1}(A_1 A_2)} Kw.$$

The first identity. Denote by C_s the line $x + is$ parallel to the x -axis. One has

$$\begin{aligned}
K(1 + q\widehat{Y})\widehat{X}w &= \int_{C_0} (1 + qe^x) w(x + 2\pi i\hbar) \Phi^\hbar(x) e^{-xz/2\pi i\hbar} dx \\
&\stackrel{(3)}{=} \int_{C_0} w(x + 2\pi i\hbar) \Phi^\hbar(x + 2\pi i\hbar) e^{-xz/2\pi i\hbar} dx = \int_{C_{2\pi i\hbar}} w(x) \Phi^\hbar(x) e^{-(x-2\pi i\hbar)z/2\pi i\hbar} dx
\end{aligned}$$

$$\stackrel{(*)}{=} e^z K w = \widehat{Y} K w.$$

To obtain the equality $(*)$ we have to move the contour $C_{2\pi i\hbar}$ down to C_0 . We can justify this because: (i) the function $\Phi^{\hbar}(z)$ is analytic in the upper half plane, see (2), and (ii) the function $\Phi^{\hbar}(z)$ growth on any horizontal line at most exponentially, while $w(x)$ decays there much faster, like e^{-x^2} .

Remark. We used here $\hbar > 0$. We would not be able to move a similar contour with negative imaginary part, since it will hit the poles of $\Phi^{\hbar}(z)$.

The second identity.

$$K\widehat{Y}^{-1}w(z) = K e^{-x} w = \int_{-\infty}^{\infty} w(x) \Phi^{\hbar}(x) e^{-x(z+2\pi i\hbar)/2\pi i\hbar} dx = \widehat{X} K w(z).$$

The third identity. We have

$$\begin{aligned} & \int_{C_0} w(x - 2\pi i\hbar) \Phi^{\hbar}(x) e^{-xz/2\pi i\hbar} dx \\ &= \int_{C_{-2\pi i\hbar}} w(x) \Phi^{\hbar}(x + 2\pi i\hbar) e^{-(x+2\pi i\hbar)z/2\pi i\hbar} dx \\ &= e^{-nz} \int_{C_{-2\pi i\hbar}} w(x) \Phi^{\hbar}(x + 2\pi i\hbar) e^{-xz/2\pi i\hbar} dx. \end{aligned}$$

We can move the contour $C_{-2\pi i\hbar}$ up towards C_0 since the function $\Phi^{\hbar}(x + 2\pi i\hbar)$ is holomorphic above the line $C_{-2\pi i\hbar}$, and grows in horizontal directions in the area between the two contours at most exponentially, while $w(x)$ decays like e^{-x^2} . So we get

$$\begin{aligned} & e^{-z} \int_{C_0} w(x) \Phi^{\hbar}(x + 2\pi i\hbar) e^{-xz/2\pi i\hbar} dx \\ &= e^{-z} \int_{C_0} w(x) \Phi^{\hbar}(x) (1 + qe^x) e^{-xz/2\pi i\hbar} dx \\ &= Y^{-1} (1 + q\widehat{X}^{-1}) \int_{C_0} w(x) \Phi^{\hbar}(x) e^{-xz/2\pi i\hbar} dx. \end{aligned}$$

Lemma 2.8 is proved.

To show that $Ks \in S_{\mathbf{L}}$ for an $s \in S_{\mathbf{L}}$ we need to check that for any $B \in \mathbf{L}$ the functional $w \rightarrow (Ks, \widehat{B}^* w)$ is continuous. Since W is dense in $S_{\mathbf{L}}$ by Theorem 2.3, there is a sequence $v_i \in W$ converging to s in $S_{\mathbf{L}}$. This means that

$$\lim_{i \rightarrow \infty} (\widehat{B} v_i, w) = (\widehat{B} s, w) \quad \text{for any } B \in \mathbf{L}, w \in W. \quad (16)$$

One has

$$\begin{aligned} (Ks, \widehat{B}^* w) &= (s, K^{-1} \widehat{B}^* w) \stackrel{\text{Prop. 2.7}}{=} (s, \widehat{\gamma(B^*)} K^{-1} w) \stackrel{(16)}{=} \lim_{i \rightarrow \infty} (v_i, \widehat{\gamma(B^*)} K^{-1} w) \\ &\stackrel{\text{def}}{=} \lim_{i \rightarrow \infty} (\widehat{\gamma(B)} v_i, K^{-1} w) \stackrel{(16)}{=} (\widehat{\gamma(B)} s, K^{-1} w) = (K \widehat{\gamma(B)} s, w). \end{aligned}$$

Since the functional on the right is continuous, $Ks \in S_{\mathbf{L}}$, and we have (13). The theorem is proved.

Since $\gamma^5 = \text{Id}$, Theorem 2.6 immediately implies

Corollary 2.9 *For any $A \in \mathbf{L}$ one has $K^{-5} \widehat{A} K^5 = \widehat{A}$ on S .*

2.5 Proof of Theorem 1.1

Let $E = \{f \in L^2(\mathbb{R}) | e^{nx} f(x) \in L^2(\mathbb{R}) \text{ for any } n > 0\}$.

Lemma 2.10 $K^5(E) \subset E$.

Proof. Indeed, since $\widehat{Y} = e^x$ and $Y^n \in \mathbf{L}$ for any $n > 0$, one has $K^5 e^{nx} f = e^{nx} K^5 f$ for any $n > 0$, $f \in S$ by Corollary 2.9. So using the Remark in the end of Section 2.3, we see that W , and hence S is dense in E , we get the claim.

Lemma 2.11 K^5 is the operator of multiplication by a function $F(x)$.

Proof. We claim that the value $(K^5 f)(a)$ depends only on the value $f(a)$. Indeed, for any $f_0(x) \in E$, $f_0(a) = f(a)$ we have $f = (e^x - e^a)\phi(x) + f_0(x)$, where $\phi(x) = (f - f_0)/(e^x - e^a) \in E$. Thus $K^5 f = (e^x - e^a)K^5 \phi(x) + K^5 f_0(x)$. So $K^5 f(a) = K^5 f_0(a)$. Now define $F(a)$ from $K^5 f_0(a) = F(a)f_0(a)$. The lemma is proved.

Proposition 2.12 The function $F(z)$ is a constant.

Proof. Let S_1 be the common domain of definition of the operators $\widehat{X}^a \widehat{Y}^b$, $a \leq 0, b \geq 0$.

Lemma 2.13 The space S_1 consists of the functions $f(x)$ in L_2 which admit an analytic continuation to the upper half plane $y > 0$, and decay faster than e^{ax} for any $a > 0$ on each line $x + iy$.

Proof. Indeed, it is invariant under multiplication by e^{bx} , $b > 0$, and shift by $2\pi ia$, $a > 0$, which means that Fourier transform of a function from S_1 is invariant under multiplication by e^{ax} , $a > 0$. The lemma is proved.

Since $K^5 S \subset S$, it follows that $F(x)w \in S \subset S_1$ for $w \in W$, and hence $F(z)$ is analytic in the half plane $y > 0$. The operator of multiplication by $F(z)$ commutes with the shifts by $2\pi i$ and $2\pi i\hbar$. Thus it commutes with the shift by $2\pi i(n + m\hbar)$, $m, n > 0$. This implies that $F(z)$ is invariant under the shifts by $2\pi i(m + n\hbar)$ where $m + n\hbar > 0$. Thus $F(z)$ is a constant when \hbar is irrational. Since K^5 depends continuously on \hbar , we get Proposition 2.12, and hence Theorem 1.1.

3 Algebraic geometry of $\mathcal{M}_{0,5}^{\text{cyc}}$

Recall the cross-ratio $r(x_1, x_2, x_3, x_4)$ of four points on \mathbb{P}^1 :

$$r(x_1, x_2, x_3, x_4) := \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_2 - x_3)}, \quad r(\infty, -1, 0, z) = z.$$

It satisfies the relations $r(x_1, x_2, x_3, x_4) = r(x_2, x_3, x_4, x_1)^{-1} = -1 - r(x_1, x_3, x_2, x_4)$.

Let $\mathcal{M}_{0,5}$ be the moduli space of configurations of five distinct points on \mathbb{P}^1 considered modulo the action of PGL_2 . The moduli space $\overline{\mathcal{M}}_{0,5}$ is a smooth algebraic surface compactifying $\mathcal{M}_{0,5}$. There are 10 projective lines D_{ij} , $1 \leq i < j \leq 5$, inside of $\overline{\mathcal{M}}_{0,5}$, forming “the divisor at infinity” $D = \cup D_{ij}$. The line D_{ij} parametrizes configurations of points $(x_1, x_2, x_3, x_4, x_5)$ where “ x_i collides with x_j ”. So $\overline{\mathcal{M}}_{0,5} - D = \mathcal{M}_{0,5}$.

We picture points x_1, \dots, x_5 at the vertexes of an oriented pentagon, whose orientation agrees with the cyclic order of the points. Given a triangulation T of the pentagon, let us define a pair of rational functions on the surface $\overline{\mathcal{M}}_{0,5}$, assigned to the diagonals of the triangulation. Given a diagonal E , let z_1, z_2, z_3, z_4 be the configuration of four points at the vertexes of the rectangle containing E as a diagonal, so that z_1 is a vertex of E . Then we set

$$X_E^T := r(z_1, z_2, z_3, z_4).$$

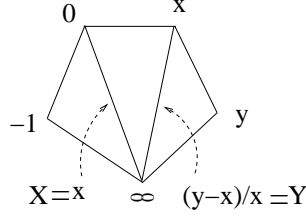


Figure 3:

Example. Given a configuration $(\infty, -1, 0, x, y)$, and taking the triangulation related to the vertex at ∞ , we get functions $X = x, Y = (y - x)/x$, see Fig. 1.1.

Definition 3.1 $\mathcal{M}_{0,5}^{\text{cyc}} := \overline{\mathcal{M}}_{0,5} - \cup_{c=1}^5 D_{c,c+1}$, where c is modulo 5.

The space $\mathcal{M}_{0,5}^{\text{cyc}}$ is determined by a choice of cyclic order of configurations of points (x_1, \dots, x_5) . Let us define embeddings $\psi_c : \mathbb{C}^* \times \mathbb{C}^* \hookrightarrow \overline{\mathcal{M}}_{0,5}$ for $c \in \{1, \dots, 5\}$. Set

$$\psi_1 : (X, Y) \longmapsto (\infty, -1, 0, X, X(1 + Y)).$$

One easily checks that it is an embedding. The map ψ_c is obtained from ψ_1 by the cyclic shift of the configuration of five points by $2c$. So it is also an embedding.

The following function is regular on the surface $\mathcal{M}_{0,5}^{\text{cyc}}$:

$$X_{a,b;c} := r(x_c, x_{c+1}, x_{c+2}, x_{c+3})^a r(x_c, x_{c+2}, x_{c+3}, x_{c+4})^b, \quad a \geq 0, b \leq 0.$$

Indeed, the poles of the first factor are at the divisor $D_{c,c+1} \cup D_{c+2,c+3}$, and the poles of the second one are at the divisor $D_{c+2,c+3} \cup D_{c,c+4}$. The set of functions $\{X_{a,b;c}\}$ coincides with the one defined in Section 2.2. Indeed, one checks this for $c = 1$ using Fig. 3, and use equivariance with respect to the shifts and Fig. 4.

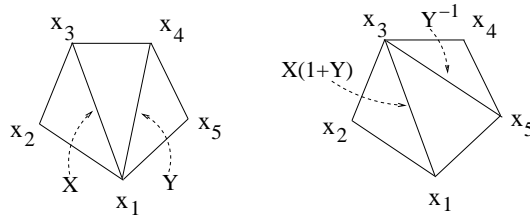


Figure 4: Change of the coordinates under a flip.

Theorem 3.2 (i) The surface $\mathcal{M}_{0,5}^{\text{cyc}}$ is the union of the five open subsets $\psi_c(\mathbb{C}^* \times \mathbb{C}^*)$ in $\overline{\mathcal{M}}_{0,5}$.

(ii) The functions $X_{a,b;c}$, where $a, b \in \mathbb{Z}, a \geq 0, b \leq 0$ and c is mod 5 form a basis of the space of regular functions on the surface $\mathcal{M}_{0,5}^{\text{cyc}}$.

Proof. (i) Straightforward.

(ii) The algebra of regular functions on $\mathcal{M}_{0,5}$ is defined as follows. Take the configuration space $\text{Conf}_5(V_2)$ of 5-tuples of vectors (v_1, \dots, v_5) in generic position in a two-dimensional symplectic vector space V_2 , modulo the SL_2 -action. The group $H := (\mathbb{C}^*)^5$ acts on it by multiplying each vector v_i by a number λ_i . Let Δ_{ij} is the area in V_2 of the parallelogram $\langle v_i, v_j \rangle$. Then

$$\mathbb{Z}[\mathcal{M}_{0,5}] = \mathbb{Z}[\text{Conf}_5(V_2)]^H = \mathbb{Z}[\Delta_{ij}^{\pm 1}]^H. \quad (17)$$

The subspace $\mathbb{Z}[\mathcal{M}_{0,5}^{\text{cyc}}]$ is spanned by the monomials

$$\prod_{1 \leq i < j \leq 5} \Delta_{ij}^{a_{ij}}, \quad (18)$$

where $a_{i,i+1} \in \mathbb{Z}$, and $a_{ij} \in \mathbb{Z}_{\geq 0}$ unless $j = i \pm 1 \pmod{5}$. Write the integers a_{ij} on the diagonals and sides of the pentagon. Call them the *weights* and the corresponding picture the *chord diagram*. The H -invariance means that the sum of the weights assigned to the edges and sides sharing a vertex is 0. We erase diagonals of weight 0. A monomial is *regular* if its chord diagram has no intersecting diagonals. Using the Plücker relations $\Delta_{ac}\Delta_{bd} = \Delta_{ab}\Delta_{cd} + \Delta_{ad}\Delta_{bc}$, $1 \leq a < b < c < d \leq 5$, and arguing by induction on the sum of the products of the weights of diagonals in the intersection points, we reduce any sum of monomials (18) to a sum of the regular ones. An easy argument with the “sum of the weights at a vertex equals zero” equations shows that for a regular monomial there exists a vertex of the pentagon such that its weights are as on Fig. 5. So the functions $X_{a,b;c}$ span the space of regular functions on $\mathcal{M}_{0,5}^{\text{cyc}}$. To check that they are linearly independent, look at the monomials with maximum value of $a + b$. The theorem is proved.

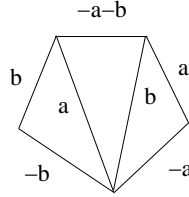


Figure 5: The weight diagram of a basis monomial; $a, b \geq 0$.

The quantized $\mathcal{M}_{0,5}^{\text{cyc}}$ at roots of unity. Assume that q is a primitive N -th root of unity. Then the functions $x_{a,b;c} := X_{Na,Nb;c}$ generate the center of the algebra \mathbb{L}_q . In particular $x := X^N, y := Y^N$ are in the center. One checks ([FG2], Section 3) that the elements x, y behave under flips just like the corresponding coordinates on $\mathcal{M}_{0,5}^{\text{cyc}}$. Therefore the spectrum of the center of \mathbb{L}_q is identified with $\mathcal{M}_{0,5}^{\text{cyc}}$. Restricting to an affine chart of $\mathcal{M}_{0,5}^{\text{cyc}}$ with coordinates (α, β) we see that the localization of the algebra \mathbb{L}_q at this chart is identified with the algebra generated by X, Y with the relations $X^N = \alpha, Y^N = \beta, XY = q^2 YX$. It is well known that it is a sheaf of central simple algebras over $\mathbb{C}^* \times \mathbb{C}^*$. So we get

Proposition 3.3 *Let q be a root of unity. Then the algebra \mathbb{L}_q gives rise to a sheaf of Azumaya algebras on $\mathcal{M}_{0,5}^{\text{cyc}}$.*

The real positive part of $\mathcal{M}_{0,5}^{\text{cyc}}$ is given by configurations of points $(\infty, -1, 0, x, y)$ with $0 < x < y$. Its closure in $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ is the pentagon. Its sides are real segments on the divisors $D_{c,c+1}$.

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